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Integrable stimulated Raman scattering systems with damping

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Abstract. We construct stimulated Raman scattering systems with damping which can be solved by the inverse scattering transform with a variable spectral parameter, and present the corresponding linear systems. We also analyse the non-standard properties of the damped soliton solution; amplitude, speed, and form change as the soliton propagates through a medium.

1. Introduction

The soliton equations, integrable by the inverse scattering transform (IST), are conservative equations. Even small damping destroys the integrability. However, the IST with a variable spectral parameter [1], which is a generalization of the traditional IST, allows one to study integrable systems in the presence of special forms of damping of arbitrary strength. We find that there are special forms of damping which allow an integrable system to remain integrable. We call these special forms of damping 'integrable damping'. They can be regularly found by the IST with a variable spectral parameter. What we do here is to apply the method of the variable spectral parameter to the general stimulated Raman scattering (SRS) and two-photon propagation (TPP) equations [2]. We find that there are several interesting forms of integrable damping which allow the resulting equations to remain integrable.

Let us first review the basic undamped SRS/TPP equations. They are [2]

$$\begin{aligned} \partial_\tau r &= \mathrm{i}(g r s_3 + r_3 s) \qquad & \partial_\tau r_3 &= \tfrac{1}{2}\mathrm{i}(r\bar{s} - \bar{r}s) \\ \partial_\chi s &= \mathrm{i}(g s r_3 + \epsilon s_3 r) \qquad & \partial_\chi s_3 &= \tfrac{1}{2}\mathrm{i}(s\bar{r} - \bar{s}r) \end{aligned} \tag{1}$$

where $\epsilon = +1$ is used for SRS, and $\epsilon = -1$ for TPP. The system is written in characteristic coordinates τ and χ, τ is the retarted time

$$\tau = t - x/v$$

where $\chi = x$ is the spatial coordinate, v is a group velocity of electromagnetic waves and g is a dynamical Stark shift coefficient. The Stokes vector $s = (\mathrm{Re}\, s, \mathrm{Im}\, s, s_3)$ characterizes the pump and Stokes electromagnetic waves propagating through an optical media, represented by the Bloch vector $r = (\mathrm{Re}\, r, \mathrm{Im}\, r, r_3)$, $s_3 = I_\mathrm{p} - I_\mathrm{s}$ is the difference between the intensities of pump- and Stokes-beams; r_3 is the inversion corresponding to the forbidden transition between ground and upper levels. Equations (1) can be integrated by inverse scattering

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transform (IST) with a constant spectral parameter. The corresponding Lax pair for (1) is the following [3–5]:

$$\Phi_\tau + U\Phi = 0 \tag{2a}$$

$$\Phi_\chi + V\Phi = 0 \tag{2b}$$

$$U = \lambda u_1 + u_0 \qquad V = v_0 + \frac{v_1}{\lambda + \frac{1}{2}g} \tag{3}$$

$$u_1 = \begin{pmatrix} \mathrm{i}s_3 & -s \\ \epsilon\bar{s} & -\mathrm{i}s_3 \end{pmatrix} \qquad v_1 = -\frac{1}{2}\begin{pmatrix} -\mathrm{i}\epsilon r_3/2 & (\mathrm{i}f - \frac{1}{2}g)\epsilon r \\ (\mathrm{i}f + \frac{1}{2}g)\bar{r} & \mathrm{i}\epsilon r_3/2 \end{pmatrix}$$

$$u_0 = -\mathrm{i}f\begin{pmatrix} 0 & s \\ \epsilon\bar{s} & 0 \end{pmatrix} \qquad v_0 = -\frac{1}{2}\begin{pmatrix} \mathrm{i}gr_3 & \epsilon r \\ -\bar{r} & -\mathrm{i}gr_3 \end{pmatrix}$$

where the bars refer to complex conjugation. Here $f^2 = (\epsilon - g^2)/4$ and λ is a constant spectral parameter.

The main goal of this work is to construct new integrable SRS models that include damping. Of these forms of integrable damping which we have found, two interesting ones are

$$\begin{aligned} \partial_\tau r &= \mathrm{i}(grs_3 + r_3 s) - \tilde{c}r & \partial_\tau r_3 &= \tfrac{1}{2}\mathrm{i}(r\bar{s} - \bar{r}s) - \tilde{c}r_3 \\ \partial_\chi s &= \mathrm{i}(gsr_3 + s_3 r) & \partial_\chi s_3 &= \tfrac{1}{2}\mathrm{i}(s\bar{r} - \bar{s}r) \end{aligned} \tag{4}$$

$$\begin{aligned} \partial_\tau r &= \mathrm{i}(grs_3 + r_3 s) & \partial_\tau r_3 &= \tfrac{1}{2}\mathrm{i}(r\bar{s} - \bar{r}s) \\ \partial_\chi s &= \mathrm{i}(gsr_3 + s_3 r) - as & \partial_\chi s_3 &= \tfrac{1}{2}\mathrm{i}(s\bar{r} - \bar{s}r) - as_3 \end{aligned} \tag{5}$$

where $\tilde{c}, a =$ real constants. We remark that (4) and (5) can only be obtained when

$$\epsilon = 1 \qquad g^2 = 1 \tag{6}$$

which means that $f = 0$. In other words, (4) and (5) are only integrable for the SRS case and only when the Stark coefficient has a special value. This is not surprising and frequently occurs in other systems when one demands integrability.

Let us now make some remarks on the variable spectral parameter method [1]. Equations (2) and (3) are actually an overdetermined set of linear equations with a spectral parameter. The condition necessary for a solution to exist (integrability condition) is (1), provided λ is a constant. However, there is no reason for λ to be a constant except for simplicity. It was proposed in [1] to consider λ as a function of time, coordinate and an additional complex constant, called the 'hidden' spectral parameter. The proposed method was coined as the IST with a variable spectral parameter. For each soliton equation, integrable by the traditional IST, one can generate the whole class of new equations, integrable by IST with a variable spectral parameter. The elements of this class are called 'deformations' of the original soliton system. So, systems (4) and (5) are 'deformations' of the SRS system (1). To construct special solutions, e.g. solitons, for 'deformations' one can use, among other things, the 'dressing' technique that was developed originally for constant λ [6]. The new solitons behave in new and interesting fashions. All of the soliton's characteristics such as amplitude, speed, and form change as the soliton propagates through a media.

The method proposed in [1] is a development of ideas of [8–10]. A series of 'deformations' was also obtained in [11–14]; a symmetry approach was designed in [15]. The method to produce the finite-gap solutions was developed in [7].

The outline of this paper is as follows: in section 2, we analyse the most general deformation of the SRS system (1). Here we find that the general case naturally splits into several subclasses. We then analyse all these subclasses. They are each a special deformation of the general case. We also present the integrable equations (with integrable damping) for each subclass.

We discuss and describe how a known [18] integrable case of SRS with conductivity fits into our scheme. This case is an example of a 'trivial' deformation. We show that it is nothing more than a special transformation of independent and dependent variables. Thus the point here is that sometimes the variable spectral method does not generate a genuinely new system, but rather simply generates a transformation of an integrable system.

However, this does not always happen. As an example, the most interesting of these subclasses is (4), as it corresponds to the case where the atomic equilibrium for the atoms is fifty percent excited. Such a situation could be achieved by suitable pumping. Then if the two relaxation times (T, T_3) were equal, equation (4) would be the appropriate equation. Thus the soliton solutions of this equation would be of interest. But we find that they are also quite interesting in their own right, since they are uniquely different from other solitons in several ways. We also show that this is a non-trivial deformation in that there can be no transformation of independent and dependent variables which will reduce this system to an integrable undamped system.

2. Inverse scattering transform with a variable spectral parameter

2.1. General deformation of SRS (TPP) system

Now we shall use a variable spectral parameter to deform the system (1). To do so, we only need to make one modification in a traditional IST. λ is a function of τ, χ and some 'hidden' spectral parameter z. The dependence on z is important as it allows the functions λ and $1/(\lambda + \frac{1}{2}g)$ to be linear-independent. As we shall see later, the function $\lambda = \lambda(\tau, \chi, z)$ is not fully arbitrary.

If equation (2) is to be integrable, then it follows that

$$\frac{\partial U}{\partial \chi} - \frac{\partial V}{\partial \tau} = [U, V]. \tag{7}$$

Evaluation of (7), when λ is a constant parameter, generates an expression with terms proportional to λ, 1 and $(\lambda + \frac{1}{2}g)^{-1}$. If λ is variable, then the derivatives of λ should not generate additional functions of λ, so we must take

$$\frac{\partial \lambda}{\partial \chi} = a\lambda + b + \frac{c}{\lambda + \frac{1}{2}g} \tag{8a}$$

$$\frac{\partial}{\partial \tau}\frac{1}{\lambda + \frac{1}{2}g} = \tilde{a}\lambda + \tilde{b} + \frac{\tilde{c}}{\lambda + \frac{1}{2}g} \tag{8b}$$

where $a, b, c, \tilde{a}, \tilde{b}$ and $\tilde{c}$ are arbitrary coefficients to be determined. This gives us a closed system for the function $\lambda = \lambda(\tau, \chi, z)$. Now using (8) in (7) gives three matrix nonlinear equations

$$\partial_\tau v_1 + \left[u_0 - \tfrac{1}{2}gu_1, v_1\right] = -\tilde{c}v_1 + cu_1 \tag{9a}$$

$$\partial_\chi u_1 + [v_0, u_1] = -au_1 + \tilde{a}v_1 \tag{9b}$$

$$\partial_\chi u_0 - \partial_\tau v_0 - [u_0, v_0] - [u_1, v_1] = -bu_1 + \tilde{b}v_1 . \tag{9c}$$

Rewriting (9*a*) by component gives equations for r and r_3

$$\begin{aligned}\partial_\tau r &= \mathrm{i}(grs_3 + r_3 s) - \tilde{c}r - 8c\left(\mathrm{i}f + \tfrac{1}{2}g\right)s \\ \partial_\tau r_3 &= \tfrac{1}{2}\mathrm{i}(r\bar{s} - \bar{r}s) - \tilde{c}r_3 + 4cs_3 .\end{aligned} \tag{9a'}$$

Doing the same with (9*b*) gives

$$\begin{aligned}\partial_\chi s &= \mathrm{i}(gsr_3 + \epsilon s_3 r) - as + \tfrac{1}{2}\tilde{a}\left(\mathrm{i}f - \tfrac{1}{2}g\right)\epsilon r \\ \partial_\chi s_3 &= \tfrac{1}{2}\mathrm{i}(s\bar{r} - \bar{s}r) - as_3 + \tfrac{1}{4}\epsilon\tilde{a}r_3 .\end{aligned} \qquad (9b')$$

The system (9a') and (9b') represents a general class of deformations of SRS (TPP) system. *If λ is constant, then the third matrix equation (9c) generates no additional equations for* r, r_3, s and s_3 provided (9*a*) and (9*b*) are true. When λ is not a constant, then (9*c*) generates, in general, additional equations and causes the nonlinear system to be overdetermined. To avoid the overdeterminacy and to avoid constraining the r and s fields, one may assume additional constraints on coefficients $a, b, c, \tilde{a}, \tilde{b}$ and $\tilde{c}$ which have so far been arbitrary. From (9*c*), one finds that the first set of these constraints must be

$$b = -2gc \qquad \tilde{b} = -2\epsilon g\tilde{c} . \qquad (10)$$

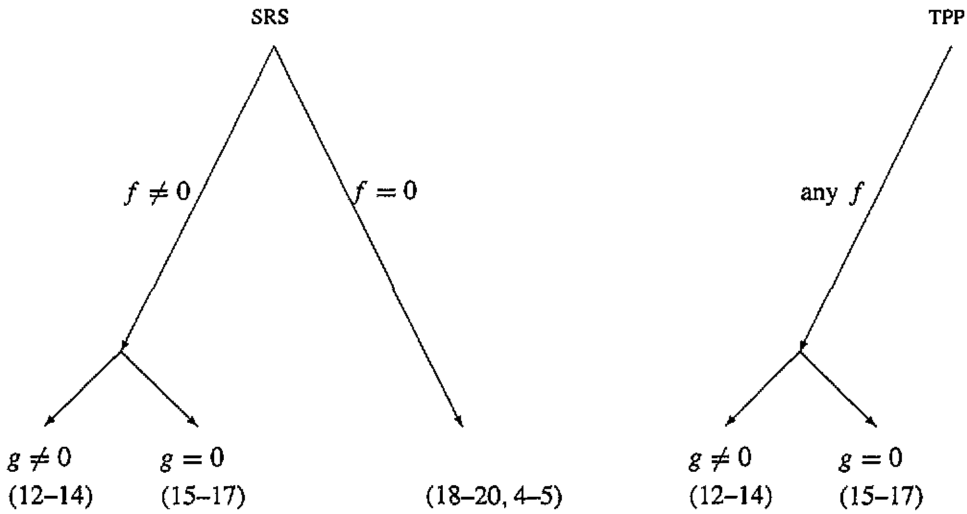
For the other constraints contained in (9*c*), there are two possibilities. If it happens that $f = 0$, then (9*c*) automatically holds true, provided (9*a*) and (9*b*) are satisfied. Then in this case there would be no further constraints other than (10) ($f = 0$ only if $\epsilon = +1$ (SRS) and $g^2 = 1$). But if $f \neq 0$, then there will be a further set of constraints. These are

$$c = \frac{\frac{1}{4}\mathrm{i}fa}{\epsilon\left(\mathrm{i}f + \frac{1}{2}g\right) - \frac{1}{2}g} \qquad \tilde{c} = \tfrac{1}{4}\epsilon\tilde{a} . \qquad (11)$$

Now there is only one other thing to consider. At this point we have equations (2) integrable, provided (8) is also integrable. As noted above, there are two cases. First, when (i) $f = 0$ and (10) must hold and second, when (ii) $f \neq 0$ and (10) and (11) must hold. So, we still must insure that (8) is also integrable, subject to the algebraic constraints for these two subcases. We shall consider these integrability conditions for (8) for each of the above subcases below.

2.2. *Specific equations*

It is convenient to summarize all following results on specific deformations of the SRS (TPP) system in a graphical tree form:



We will take up each one of these branches individually. First, for the left branch of either the SRS or TPP tree we always have $f \neq 0$ with both (10) and (11) being necessary. Then integrability of (8) along with (10) and (11) gives two possible solutions depending on whether or not $g = 0$. If $g \neq 0$, then the integrability of (8) only requires

$$a\tilde{a} = 0 \tag{12}$$

so that either a or $\tilde{a}$ must be zero. The corresponding nonlinear systems are, for $\tilde{a} = 0$

$$\begin{aligned}\partial_\tau r &= \mathrm{i}(grs_3 + r_3 s) - 8c(\mathrm{i}f + \tfrac{1}{2}g)s \\ \partial_\tau r_3 &= \tfrac{1}{2}\mathrm{i}(r\bar{s} - \bar{r}s) + 4cs_3\end{aligned} \tag{13}$$

$$\begin{aligned}\partial_\chi s &= \mathrm{i}(gsr_3 + \epsilon s_3 r) - as \\ \partial_\chi s_3 &= \tfrac{1}{2}\mathrm{i}(s\bar{r} - \bar{s}r) - as_3\end{aligned}$$

and for $a = 0$

$$\begin{aligned}\partial_\tau r &= \mathrm{i}(grs_3 + r_3 s) - \tfrac{1}{4}\epsilon\tilde{a}r \\ \partial_\tau r_3 &= \frac{2}{\mathrm{i}}(r\bar{s} - \bar{r}s) - \tfrac{1}{4}\epsilon\tilde{a}r_3\end{aligned} \tag{14}$$

$$\begin{aligned}\partial_\chi s &= \mathrm{i}(gsr_3 + \epsilon s_3 r) + \tfrac{1}{2}\epsilon\tilde{a}(\mathrm{i}f - \tfrac{1}{2}g)r \\ \partial_\chi s_3 &= \tfrac{1}{2}\mathrm{i}(s\bar{r} - \bar{s}r) + \tfrac{1}{4}\epsilon\tilde{a}r_3 .\end{aligned}$$

Here a and $\tilde{a}$ may be not only constants, but also any functions of χ and τ, respectively. Note that (13) and (14) involve a sort of 'cross-damping' where $s(s_3)$ drives $r(r_3)$ in (13). The same is true in (14) except that r and s are interchanged.

If $g = 0$, then the integrability of (8) requires

$$a_\tau + \epsilon a\tilde{a}/2 = 0 \qquad \tilde{a}_\chi + 2\tilde{a}a = 0 . \tag{15}$$

Of course, this would be the same as (12) if we took a and $\tilde{a}$ to be constant. However, (15) allows a more general class of solutions. Applying the substitution

$$a = \tfrac{1}{2}\epsilon(\ln G)_\chi \qquad \tilde{a} = 2(\ln G)_\tau$$

reduces (15) to a wave equation for G.

$$G_{\tau\chi} = 0 . \tag{16}$$

The related nonlinear system is now

$$\begin{aligned}\partial_\tau r &= \mathrm{i}r_3 s - \tfrac{1}{4}\epsilon\tilde{a}r - \mathrm{i}\epsilon^{1/2}as \\ \partial_\tau r_3 &= \tfrac{1}{2}\mathrm{i}(r\bar{s} - \bar{r}s) - \tfrac{1}{4}\epsilon\tilde{a}r_3 + \epsilon as_3 .\end{aligned} \tag{17}$$

$$\begin{aligned}\partial_\chi s &= \mathrm{i}\epsilon s_3 r - as + \tfrac{1}{4}\mathrm{i}\epsilon^{1/2}\tilde{a}r \\ \partial_\chi s_3 &= \tfrac{1}{2}\mathrm{i}(s\bar{r} - \bar{s}r) - as_3 + \tfrac{1}{4}\epsilon\tilde{a}r_3 .\end{aligned}$$

and the branch for $\epsilon^{1/2}$ is arbitrary.

Note that we now have a more general type of 'cross-damping' whereby both $r(r_3)$ and $s(s_3)$ drives the other one. Thus we have a more general linear coupling between the fields.

Now, all that is left is the right branch of SRS tree. In this case, when $f = 0$, (6) and (10) are true, we obtain a more general nonlinear system for the coefficients

$$\begin{aligned}&\tilde{a}_\chi + 2a\tilde{a} = 0 \qquad &&\tilde{c}_\chi + a\tilde{c} + \tfrac{1}{2}a\tilde{a} + 3\tilde{a}c = 0 \\ &c_\tau + 2\tilde{c}c = 0 \qquad &&a_\tau + \tilde{c}a + 8c\tilde{c} + 3\tilde{a}c = 0 .\end{aligned} \tag{18}$$

At the moment, there is no general solution of (18), and it is not clear if there are interesting cases with this general structure. However, there are interesting subcases contained in the seemingly trivial solution

$$\tilde{a} = 0 = c \tag{19}$$

which reduces (18) to

$$\tilde{c}_\chi + a\tilde{c} = 0 \qquad a_\tau + \tilde{c}a = 0\,. \tag{20}$$

If one takes $\tilde{c} = (\ln G)_\tau$, $a = (\ln G)_\chi$, then (20) is transformed into (16).

Particular simple examples follow from (20) upon taking $a = 0$ and $\tilde{c} \neq 0$ in which case, the general system (9) reduces to (4) which is SRS system with a damping that affects only the optical media. If we choose $a \neq 0$ and $\tilde{c} = 0$, then (9) reduces to (5). In this case, the electromagnetic wave is damped by conductivity. We remark that the special deformations (13) and (14) ($f \neq 0, g \neq 0$) of SRS (TPP) were first constructed in [14].

2.3. *The SRS equation with conductivity*

There is a very interesting case of integrable damping for the SRS equations which was obtained in [18] (see (22*a*) below). This integrable damping is simply an ordinary conductivity which at first does not seem to fit into our scheme. However, with the proper asymptotics, we can show that [18] does fit into our scheme.

First, assume that $g = 0$ and consider the very special case of the general deformation of SRS (9a') and (9b'): $\tilde{a} = 0 = \tilde{b} = \tilde{c}; 4c = a =$ constant. Second, let us scale every dependent and independent variable in the deformation as follows:

$$\begin{aligned} &\chi = \underline{\chi}/\varepsilon \qquad s = \varepsilon \underline{s} \qquad s_3 = \varepsilon \underline{s}_3 \\ &\tau = \underline{\tau} \qquad\quad r = \varepsilon \underline{r} \qquad r_3 = -1 + \mathrm{O}(\varepsilon^2) \\ &a = \varepsilon \underline{a} \end{aligned} \tag{21}$$

and then take the limit: $\varepsilon \to 0$. The bar under the variable denotes a new variable and should not be confused with the bar over the variable, which means complex conjugation. The physical meaning of this is that almost all atoms in the optical media are initially in the ground state, while low-amplitude, long-wave optical pulses are propagating through the media. Although the original system (9a') and (9b') does not seem to really apply to optics, because of the 'cross-damping' terms, nevertheless, the limiting procedure (21) leads to the following physically meaningful system, the SRS system with an ordinary conductivity $\underline{a}$

$$\begin{aligned} &\partial_{\underline{\tau}}\underline{r} = -\mathrm{i}\underline{s} \\ &\partial_{\underline{\chi}}\underline{s} = \mathrm{i}\underline{s}_3 \cdot \underline{r} - \underline{a} \cdot \underline{s} \\ &\partial_{\underline{\chi}}\underline{s}_3 = \tfrac{1}{2}\mathrm{i}(\underline{s} \cdot \bar{\underline{r}} - \bar{\underline{s}} \cdot \underline{r}) - \underline{a} \cdot \underline{s}_3\,. \end{aligned} \tag{22a}$$

The Lax pair for (22*a*) can be obtained from the same limit applied to (2) and (3). This gives

$$\Phi_{\underline{\tau}} + \underline{\lambda} \cdot \underline{u}\Phi = 0 \tag{22b}$$

$$\begin{aligned} &\Phi_{\underline{\chi}} - (\mathrm{i}\sigma_3/(4\underline{\lambda}) + \underline{v})\Phi = 0 \\ &\underline{\lambda}_{\underline{\tau}} = 0 \qquad \underline{\lambda}_{\underline{\chi}} = \underline{a} \cdot \underline{\lambda} \end{aligned} \tag{22c}$$

where

$$\underline{u} = \begin{pmatrix} \mathrm{i}\underline{s}_3 & -\underline{s} \\ \bar{\underline{s}} & -\mathrm{i}\underline{s}_3 \end{pmatrix} \qquad \underline{v} = \frac{1}{2}\begin{pmatrix} 0 & \underline{r} \\ -\bar{\underline{r}} & 0 \end{pmatrix}$$

and $\underline{\lambda} = \lambda \cdot \varepsilon$. (The authors of [18] derived the Lax pair for (22) in a different way by using equivalence between system (22*a*) and the Maxwell–Bloch system [2].)

An important point is that system (22*a*), like its counterpart, the Maxwell–Bloch system with damping [1], is in a way a trivial deformation. What happens is that there is a certain transformation of dependent and independent variables which will reduce these systems with integrable damping to a system with no damping. To see this, let

$$\eta = \underline{\tau} \qquad \xi = \underline{a}^{-1}(1 - \mathrm{e}^{-\underline{a}\cdot\underline{\chi}}) \tag{23}$$
$$(\underline{s}, \underline{s_3}, \underline{r}, \underline{r_3})(\underline{\tau}, \underline{\chi}) = (S, S_3, R, R_3)(\eta, \xi)\mathrm{e}^{-\underline{a}\cdot\underline{\chi}}.$$

Then this transformation reduces the deformed system (22*a*) into the original one

$$\begin{aligned} \partial_\eta R &= -\mathrm{i}S \\ \partial_\xi S &= \mathrm{i}S_3 R \\ \partial_\xi S_3 &= \tfrac{1}{2}\mathrm{i}(S\bar{R} - \bar{S}R). \end{aligned} \tag{24}$$

This transformation has an image transformation on the Lax pair. It is possible, in part, because the spectral parameter $\underline{\lambda}$,

$$\underline{\lambda} = \underline{\lambda}(z, \underline{\chi}) = z\mathrm{e}^{-\underline{a}\cdot\underline{\chi}} \tag{25}$$

corresponding to the system (22*a*), has a very simple structure: the z-dependence is separated from $\underline{\chi}$-dependence. To construct the image transformation of (23) on the corresponding Lax pair (22*b*) and (22*c*), one proceeds as follows. First, one substitutes (25) into the linear system (22*b*) and (22*c*) and multiplies the result by $\mathrm{e}^{\underline{a}\cdot\underline{\chi}}$. Then, it becomes clear that one can make a transformation of the spatial coordinate $\underline{\chi}$,

$$\mathrm{e}^{\underline{a}\cdot\underline{\chi}} \frac{\partial}{\partial \underline{\chi}} = \frac{\partial}{\partial \xi}$$

and that one can redefine the $\underline{s}$, $\underline{s_3}$, $\underline{r}$, and $\underline{r_3}$ fields according to (23). Finally, one obtains the Lax pair for the undamped system (24) which contains z as a constant spectral parameter.

So, one can now take any known exact solution $(R, S, S_3)(\eta, \xi)$, process it with the transformation (23) and generate an exact solution for the damped system (22*a*). Note how this transformation has changed the evolution. The system with integrable damping evolves almost like the undamped system, except that the spatial coordinate has been distorted and we have to multiply fields by $\exp(-\underline{a}\underline{\chi})$ in (23). Thus as $\underline{\chi}$ becomes much larger than $\underline{a}^{-1}$, the intensity of the physical variables is vanishing, thereby cutting off the interaction.

Let us note that, in general, separability of the variable spectral parameter into functions of z and the coordinates may be inadequate for the existance of a trivial deformation. Consider the SRS system given by (5). There the variable spectral is also separable. Nevertheless, this is not sufficient for reducing (5) to the undamped system (1). The reason being, the linear system (2*b*), unlike (22*b*), has a pole at the point $\lambda = -\frac{1}{2}g$. So, after substitution $\lambda(z, \chi)$ into (2*b*) we would obtain the fraction $1/(z + \frac{1}{2}ge^{a\chi})$, a movable simple pole, unless $g = 0$. But for (5), g is a non-zero constant: $g^2 = 1$. Similarly, one cannot construct such a transformation for the damped system (4) because the z-dependence of the corresponding $\lambda = \lambda(z, \tau)$ (see (27) below) cannot be separated from τ-dependence. So, systems (4) and (5) are non-trivial deformations of the original SRS system (1).

3. Soliton solution (damped and undamped)

If one considered the most general form of SRS with realistic physical damping terms, then the most appropriate model would be

$$\begin{aligned} \partial_\tau r &= \mathrm{i}(grs_3 + r_3 s) - \frac{r}{T} \qquad & \partial_\tau r_3 &= \tfrac{1}{2}\mathrm{i}(r\bar{s} - \bar{r}s) - \frac{r_3 - r_3^0}{T_3} \\ \partial_\chi s &= \mathrm{i}(gsr_3 + s_3 r) - \sigma s \qquad & \partial_\chi s_3 &= \tfrac{1}{2}\mathrm{i}(s\bar{r} - \bar{s}r) - \sigma s_3 \end{aligned} \tag{26}$$

where T is the time of relaxation for polarization, T_3 is the time constant for decay of the inversion and σ is a phenomenological conductivity. Unfortunately, even with a variable spectral parameter, (26) is not integrable. However, we do have two special cases of (26) which can be of interest. The first case corresponds to (4) when there is no conductivity, $\sigma = 0$, and the relaxation constants T, T_3 and r_3^0 are taken to be $T = T_3$ and $r_3^0 = 0$. If it were not for r_3^0 being zero, then this would be an important physical situation. However $r_3^0 = 0$ means that the stable equilibrium would have to be 50% excited with equal populations in each level. That can only occur physically in an infinitely high temperature or if suitably pumped. The second case is when $T = \infty = T_3$ and only the conductivity, σ, is present. This case is given by (5) and would model SRS in a conducting medium and was discussed above.

Now let us turn our attention to (4) and look at the soliton solution. We derive the one-soliton formula for (4) using the 'dressing' technique [6, 16], taking into account that λ must satisfy (8). This gives

$$\lambda = \lambda(z, \tau) = \frac{z\mathrm{e}^{\tilde{c}\tau}}{1 + 2gz(1 - \mathrm{e}^{\tilde{c}\tau})}\,. \tag{27}$$

We have also used $a = 0 = b = c = \tilde{a}, \tilde{b} = -2g\tilde{c}, \tilde{c} =$ real constant, $g = \pm 1$. Note that z is the 'hidden' spectral parameter which is, in fact, just an integration constant for the system (8). Omitting standard calculations, we present the final formulae:

$$\begin{aligned} \frac{s_3}{s_0} &= 1 - 2\frac{\lambda_{1i}^2}{|\lambda_1|^2}\frac{1}{\cosh^2\theta} \\ \frac{s}{s_0} &= 2\frac{\lambda_{1i}}{|\lambda_1|^2}\frac{\mathrm{e}^{\mathrm{i}q}}{\cosh\theta}[-\lambda_{1r} + \mathrm{i}\lambda_{1i}\tanh\theta]\,. \end{aligned} \tag{28}$$

In the above, we have taken

$$z = z_1 = \xi + \mathrm{i}\eta$$

where

$$\lambda_1 = \lambda(z = z_1, \tau) = \lambda_{1r} + \mathrm{i}\lambda_{1i}\,.$$

Note that $\lambda(\tau = 0, z) = z$, so z is also the initial value of variable spectral parameter. Also

$$\begin{aligned} \theta &= \frac{gs_0}{\tilde{c}}\arg\varphi - \frac{r_0\Delta\eta/2}{|z_1 + \frac{1}{2}g|^2} + \theta_0 \\ q &= \frac{gs_{30}}{\tilde{c}}\ln|\varphi| + r_0\Delta\left\{g - \frac{\xi + \frac{1}{2}g}{|z_1 + \frac{1}{2}g|^2}\frac{1}{2}\right\} + q_0 \end{aligned} \tag{29}$$

where θ_0 and q_0 are real constants that define the real and complex phases of the soliton, respectively; the dependence on retarded time τ and the coordinate χ is defined in the functions $\varphi(\tau)$ and $\Delta(\chi)$:

$$\varphi(\tau) = 1 + 2g\bar{z}_1(1 - e^{\tilde{c}\tau})$$
$$\Delta(\chi) = \frac{v}{\tilde{c}}(1 - e^{-\tilde{c}\chi/v}) . \tag{30}$$

If η goes to 0 we have the soliton vanishing, leaving us with only a 'background' solution:

$$s_3 = s_0 \qquad r_3 = r_0 \exp[-\tilde{c}(\tau + \chi/v)]$$
$$s = 0 \qquad r = 0 . \tag{31}$$

Transforming (31) into the laboratory frame, we obtain for r_3,

$$r_3 = r_0 e^{-\tilde{c}t} .$$

Soliton formulae (27)–(30) are parametrized by four real constants: s_0, r_0, θ_0, q_0 and by one complex constant: z_1. In the limit $\tilde{c} \to 0$, the 'damped' soliton (27)–(30) degenerates into the known SRS soliton [3, 5, 17]. Indeed as $\tilde{c} \to 0$, then $\lambda \to z$; $\Delta \to \chi$; $(g/\tilde{c}) \ln \varphi \to -\bar{z}_1\tau$, and

$$\theta \to s_0\eta\tau - \frac{r_0\eta\chi/2}{|z_1 + \frac{1}{2}g|^2} + \theta_0$$
$$q \to -s_0\xi\tau + r_0\chi g - \frac{1}{2}\frac{\xi + \frac{1}{2}g}{|z_1 + \frac{1}{2}g|^2} + q_0 . \tag{32}$$

This is an undamped Raman soliton as one can see upon simply substituting (32) into (28).

Although the structure of formula (28) for the 1-soliton solution is exactly the same as in the absence of damping, nevertheless the damping strongly affects the way τ and χ appear in the expression (28). Without damping, $\ln \varphi/\tilde{c}$ and $\Delta(\chi)$ are just linear functions of τ and χ, respectively. With damping, $\ln \varphi/\tau$ and Δ are nonlinear functions. It is they (together with expression (27) for $\lambda(z, \tau)$) that define how the soliton's parameters change as the soliton propagates through an optical media. In the 'future', $t \to +\infty$, expressions (28)–(30) simplify and describe a soliton of asymmetric shape that moves with the group velocity v of electromagnetic waves in a transparent media. To obtain this result, let us switch to the reference system that moves with the group velocity $x - vt = y$ and take the limit $t \to +\infty$ in (28)–(30). Then

$$\lambda = \lambda(z, y) = \frac{z e^{-\tilde{c}y/v}}{1 + 2gz(1 - e^{-\tilde{c}y/v})} \tag{33}$$

$$\varphi = \varphi(y) = 1 + 2g\bar{z}_1(1 - e^{-\tilde{c}y/v}) \tag{34}$$
$$\Delta \to \frac{v}{\tilde{c}}$$

in (28), (29). The asymptotic shape of the soliton is asymmetric. Figure 1 represents a plot of $s_3(y)$ (full curve) as $t \to \infty$. The broken curve denotes the shape of the soliton [5, 17] for SRS without any damping. In this figure, we have used the following values for the parameters

$$s_0 = r_0 = 1 \qquad g = v = 1 \qquad \tilde{c} = 0.3 \qquad \eta = 0.1 . \tag{35}$$

From equation (28)–(30) one can show that the damped soliton is almost standing still (in the laboratory reference system) when $\tau \leqslant 0$. Then, around $t = 1$, the soliton picks up

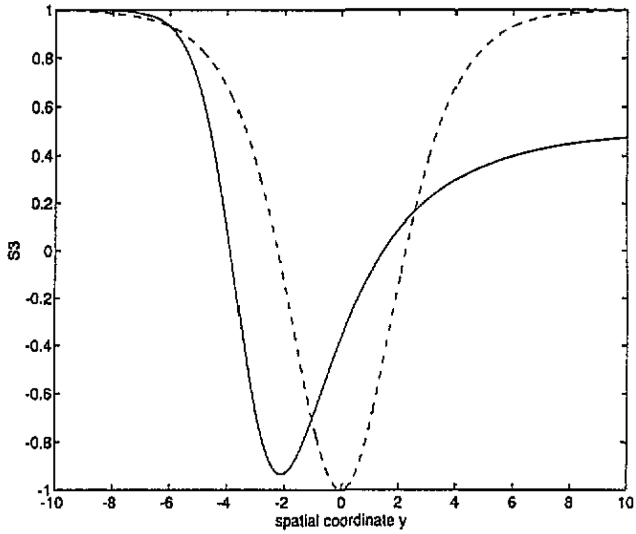


Figure 1. Damped soliton (full curve) to SRS equation (4) versus undamped soliton (broken curve).

speed, approaching the group velocity, v, of electromagnetic waves. Such a scenario may be qualitively explained by the expression

$$v_s = \frac{v}{1 + \dfrac{r_0 v/2s_0}{|z_1 + \frac{1}{2}g|^2}} \tag{36}$$

for the velocity v_s of an *undamped* Raman soliton. Here the constants r_0 and s_0 are the background values of fields r_3 and s_3, respectively. Now, switch on the damping of the type studied in this section. The first change that occurs in (36) would be that the background inversion constant, r_0, would no longer be a constant and would start to grow exponentally as $t \to -\infty$; in other words the optical medium would become too dense for the soliton to propagate. So, in this limit, $v_s \to 0$. For time going forward, $t \to +\infty$, the inversion r_0 would decay and therefore the soliton accelerates: $v_s \to v$. Figure 2 illustrates these dynamics of the damped soliton in the (x, t)-plane. (To get a better result we have plotted here the function $-s_3(x, t) + 1$ instead of $s_3(x, t)$.)

To explain other features of figure 2, we analyse expressions (27)–(29) in the limits $\tau \to \pm\infty$. If $\tau \to +\infty$; $\chi = \chi_0$, then

$$\lambda \to -\tfrac{1}{2}g \tag{37a}$$

$$\theta \to \frac{gs_0}{\bar{c}} \tan^{-1}\left[-\frac{\eta}{\xi}\right] - \frac{r_0\eta\Delta/2}{|z_1 + \frac{1}{2}g|^2} + \theta_0 \tag{37b}$$

is finite. As the result of (37*a*) and (28)

$$\lim_{\tau\to+\infty} s_3 = s_0 \,. \tag{38}$$

If we return to the laboratory reference system, the result (38) means that both limits—(i) $t \to +\infty$; $x = x_0$ and (ii) $t = t_0$; $x \to -\infty$ result in s_3 going to the constant $s_0 = 1$. If $\tau \to -\infty$; $\chi = \chi_0$, then

$$\lambda_1 \simeq \frac{z_1 e^{\bar{c}\tau}}{1 + 2gz_1}$$

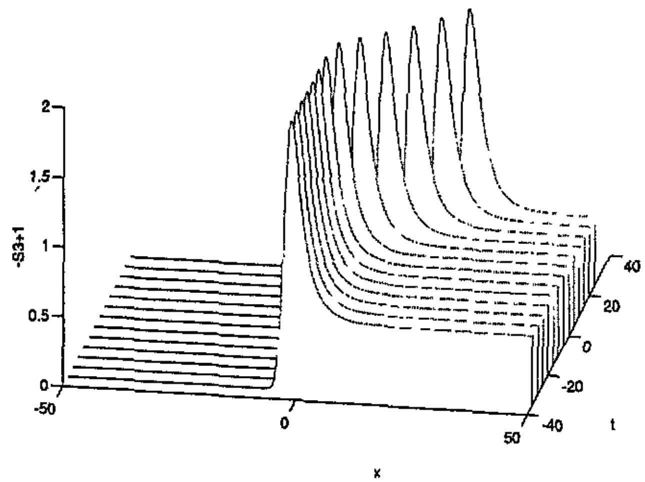


Figure 2. The dynamics of the damped soliton.

$$\frac{\lambda_{1i}^2}{|\lambda_1|^2} \to \frac{\eta^2}{|z_1|^2|1+2gz_1|^2} \tag{39a}$$

$$\theta \to \frac{gs_0}{\tilde{c}}\tan^{-1}\left[-\frac{2g\eta}{1+2g\xi}\right] - \frac{r_0\eta\Delta/2}{|z_1+\frac{1}{2}g|^2} + \theta_0 . \tag{39b}$$

Because of (39a) and (39b), $\lim s_3 \neq s_0$. Thus, both limits (i) $t \to -\infty; x = x_0$ and (ii) $t = t_0; x \to +\infty$ do not produce s_0 for s_3. (To take the latter limit, we have to take into account that $\Delta(\chi) \to v/\tilde{c}$ as $\chi \to +\infty$.) The results (39) mean that the damped soliton (28)–(30) is not really a solitary wave at all: the magnitude s_3 does not go to the same value of s_0 on the right end of the x-axis as it does on the left end. Rather it is a combination of a solitary and a shock wave. For simplicity, we present the asymptotics for the special case $\xi = 0$ only

$$\lim_{x\to+\infty}\left[\frac{s_3}{s_0} - 1\right] = -\frac{2}{1+4\eta^2}\frac{1}{\cosh^2[(gs_0/\tilde{c})\tan^{-1}(2g\eta) + [2\eta r_0 v/\tilde{c}(1+4\eta^2)] + \theta_0]} \tag{40}$$

which has an inverse dependence on the damping $\tilde{c}$. The less damping $\tilde{c}$ or the more η, the less is the right-hand side of (40) and the more the damped soliton is like an undamped one. But even for small values of the damping the 'tail' is always present.

In the absence of damping, the Raman soliton may be created in the following way. First, we have a pure ($s_0 = 1$) pump laser beam incident on an optical medium that has all atoms in the upper state ($r_0 = +1$). Such a state is a stable one, because there are no Stokes photons that would cause the atoms to decay from the upper to the ground state. Then, we switch on a Stokes beam for a certain time. As a consequence we destabilize the total configuration (pump, Stokes and medium) and atoms start to decay into the ground state, coherently emitting photons at the frequency of pump wave. After shutting off the Stokes beam and waiting for some time, there will remain a Raman soliton in the medium which is shown by the broken curve in figure 1. This optical pulse has all the regular properties of a soliton.

To create a soliton under the presence of our integrable damping $\tilde{c}$, we would probably have to change the above experimental setup in only one respect: at the beginning, we would have to use a combination of pump and Stokes incident on the optical medium (see

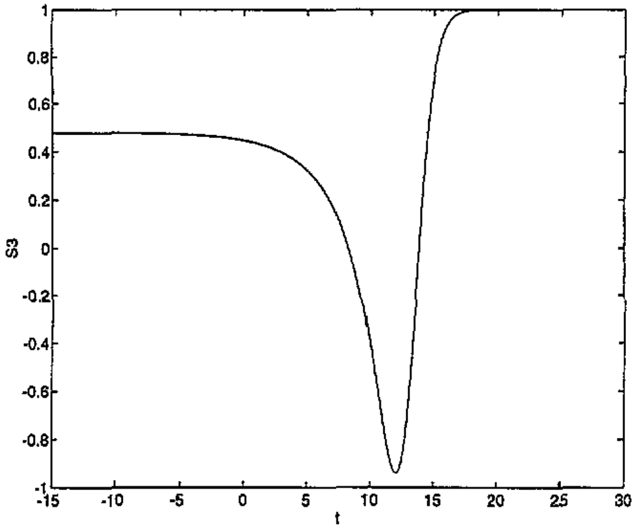


Figure 3. The soliton generating input pulse.

figure 2). The right ratio of intensities is probably determined by both the value of damping $\tilde{c}$ (40) and the amplitude of the future soliton. Then we increase the intensity of the Stokes beam for a certain time and finally turn off the Stokes beam completely. An exact profile of the damped SRS soliton is shown in figure 3. The plotted function $s_3(x = 10, t)$ represents the normalized difference between pump- and Stokes-intensities. We obtained this function from expressions (28)–(30) and (35). For convenience, we took the left edge of the optical sample to be at $x = 10$.

We suspect that there must be some other recipies for creating a soliton in the presence of integrable damping. Maybe it is not even necessary to increase the intensity of the Stokes component at all, as in the previous examples. Maybe it's sufficient just to shut off the Stokes component. To prove (or disapprove) such a scenario, it would be necessary to analyse the scattering problem ($2a$) with λ being defined by (27). This problem is *very* different from that studied before [3, 5] without any damping. The reader should note that if (27) is inserted into (3), then the analytical structure of ($2a$) will be dramatically changed. In equation ($2a$), U will have a simple pole on the real z-axis, causing Φ to have a branch-cut along the z-axis. For example, let us fix $s_3 = 1$ and $s = 0$ in ($2a$). Then the solution of ($2a$) is

$$\Phi(z, \tau) = \exp\left(\mathrm{i}\frac{\sigma_3}{2g\tilde{c}} \ln\left[1 + 2gz\left(1 - \mathrm{e}^{\tilde{c}\tau}\right)\right]\right).$$

If $\tilde{c}$ goes to zero, then $\Phi \to \exp[-\mathrm{i}z\tau\sigma_3]$ which is a regular function on a real z-axis. But if $\tilde{c} \neq 0$, then $\Phi(z, \tau)$, for any value of τ, has a branch point at

$$z = -\frac{\frac{1}{2}g}{1 - \mathrm{e}^{\tilde{c}\tau}}.$$

When τ changes from $-\infty$ to $+\infty$ the branch point sweeps out all values from $-\infty$ to $+\infty$. Thus, to define Jost functions correctly, one would have to make a cut along the real z-axis. This is in stark contrast to the undamped case where Φ only has an essential singularity at $\lambda = \infty$. Due to this branch-cut along the real z-axis, we cannot expect to be able to use any criteria from the undamped case. Rather, one would have to analyse the eigenvalue problem ($2a$) for conditions necessary and/or sufficient for the bound states to exist. So, with this in mind, it should not be too surprising if the solution for the damped case turned out to be significantly different from the undamped case. Turning our attention to (2*b*), we see that V has the very same simple pole: $\lambda = -\frac{1}{2}g$ and $z = -\frac{1}{2}g$ for both the damped and undamped cases, respectively. Consequently the evolution of the scattering data should be exactly the same in both cases.

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